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EVEN LOCAL SUBGROUPS OF A FINITE SIMPLE GROUP

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In this short article, we will have views of both the past and the future of the finite group theory. About twenty years have passed since the classification of the finite simple groups completed. New study already seems to begin vividly in some areas, quietly in others. Entering into a new century, we think it is quite important to clarify the routes that we have just traced and the ones that we may look for hereafter.

This article consists of two sections. In the first section, we briefly summarize the history of the classification of the finite simple groups. It seems very difficult (or nearly impossible) for the author to give a lecture on the whole story. Thus we place emphasis on topics on the local analysis of finite simple groups. In the second section, we would like to propose a new approach that treats the finite simple groups of characteristic 2 type using associated prime graphs.

1. Historical Note

In this section, we first have a brief review of the classification of the finite simple groups. After that, we will focus on the local subgroups of finite simple groups and appreciate some technical results.

1.1. **Existing classification.** The history of the classification begins with the affirmative solution to the Burnside conjecture. In their long and complicated paper, Feit and Thompson proved the following.

Theorem 1 (Odd Order Theorem). *Every group of odd order is solvable.*

This immediately yields that every nonsolvable finite simple group has even order. With Sylow's theorem, we can see that every finite simple group has involutions (elements of order 2).

In their earlier work, Brauer and Fowler proved the following.

Theorem 2 (Brauer Fowler Principle). *Let H be a group whose center has even order. Then there exist, if any, at most finite number of isomorphism classes of finite simple groups G such that $C_G(t) \cong H$ for some involution t of G.*

This theorem was important because it showed that we should focus on the involutions and their centralizers in order to characterize finite simple groups. Not only did they give a general principle, but also they showed an example that suggested the route we would go along with. In fact, the simple groups of Lie type of odd characteristic were characterized by centralizers of involutions with some small number of exceptions.

Pursuing the analysis of centralizers of involutions, together with extensions to centralizers of elements of odd prime order, the classification of the finite simple groups was completed in around 1980 as a joint work of Gorenstein and quite many others. The author will not mention the names of the indivisual contributors.

Theorem 3 (Classification Theorem). *Every finite simple group is isomorphic to one of the following groups: the groups of prime order, the alternating groups on five or more letters, the simple groups of Lie type over finite fields, the twenty-six sporadic groups.*

Following the Brauer Fowler principle, and making necessary extensions to that, we were able to reach the goal. Let us call the route the Gorenstein Program. In brief, it is stated in a following way.

It is centralizers of involutions that we should first focus our attention on. The actual task is divided into two parts: one is to characterize the known simple groups by centralizers of involutions, and the other is to know the structure of centralizers of involutions of finite simple groups. Most of the known simple groups (the alternating groups and the Lie type groups of odd characteristic) appear in this course. To treat the remaining groups, our attention should then go to centralizers of elements of odd prime order. By similar analysis, most of the remaining known simple groups (the Lie type groups of even characteristic) appear in this course. Summing up, we call these two courses a *semisimple* approach. Finally we have some exceptional groups which must be treated indivisually.

In the course of the classification, we discoverd twenty-six sporadic simple groups apart from the well-known alternating groups and Lie type groups. It seems still unclear why such variation can (should) exist though it becomes much clearer that they have various relations to other fields of mathematics.

In 1990s, Gorenstein, Lyons, and Solomon began to publish books toward the classification of the second generation. Let us call the route the Gorenstein, Lyons, Solomon Program.

Their project seems to have two purposes. The first is to reorganize or rearrange our knowledge, tools, and methods we already have in the existing classification. Some of them were well-known from the past and played important roles in the existing classification. Some of them were born at around the final step of the existing classification. Among such newcomers, the most fruitful is a pushing-up technique. Extending the definition of the characteristic of Lie type groups, we define a characteristic p of a finite simple group G. Focus is now on the (maximal) p local subgroups to investigate the structure of G. This is quite reasonable because most finite simple groups turn out to be of Lie type and the maximal p local subgroups are the maximal parabolic subgroups for finite simple groups of Lie type of characteristic p. We call this course a *unipotent* approach. The second purpose is to evaluates the roles of the two (simisimple and unipotent) approaches. In the existing classification, we first adopted the semisimple approach, and then the unipotent approach for the remaining (fairly small number of) groups. Either of them is not sufficient alone. However they are overlapping each other. The second part includes the investigation of the real roles of both the apporoaches.

1.2. **Local analysis.** Local analysis is one of the most important part of the classification. It is divided into two parts. One is to investigate how local subgroups of finite simple groups are constructed. The other is to investigate how local subgroups are embedded in finite simple groups.

We must first mention the remarkable work of Thompson.

Theorem 4 (N Group Theorem). *Let G be a finite simple group all of whose local subgroups are solvable. Then G is isomorphic to one of the following groups: $L_2(q)$, $Sz(q)$ (where q is a power of 2), $L_2(p)$ (where p is a Fermat or Merssenne prime with $p \geq 5$), A_6, A_7, $L_3(3)$, M_{11}, $U_3(3)$, ${}^2F_4(2)'$.*

Classification itself of such simple groups now becomes less important because the existing classification decides to use other general results. However, the methods and techniques discovered and developed in the course of the proof continue to live and apply to various scenes of local analysis.

We next mention the joint work of Janko, Smith, Gorenstein and Lyons which generalizes the N group theorem.

Theorem 5. *Let G be a finite simple group all of whose 2 local subgroups are solvable. Then G is isomorphic to one of the following groups: $L_2(q)$, $Sz(q)$, $U_3(q)$ (where q is a power of 2), $L_2(p)$ (where p is a Fermat or Merssenne prime with $p \geq 5$), A_6, A_7, $L_3(3)$, M_{11}, $U_3(3)$, ${}^2F_4(2)'$.*

This theorem is necessary for the existing classification when we investigate the "quasithin groups." We will mention later the new proof of this theorem by a pushing-up technique or a so-called "amalgam method."

1.3. **Amalgam method.** Amalgam methods (by Goldshmidt and others) were born in the end of 1970s, when the existing classification almost completed. Thus, in a fair judgment, their contribution to the existing classification is rather less than it should be.

The methods are used to investigate both the structure and the embedding of local subgroups. Finding an amalgam of subgoups in a finite simple group is almost equal to finding a set of parabolic subgoups containing a common Borel subgoup in a Lie type group. Considering the fact that most of the finite simple groups are of Lie type, it is quite natural for us to apply these methods to a new classification.

In an original form, an amalgam method is explained in the following way. There are two steps to apply to some class of finite simple groups: one is to classify the isomorphism types of the triples (X, B, Y) having properties such as (1) both $|X : B|$ and $|Y : B|$ are odd, (2) $O_B(X, Y) = 1$, and so on; the other is to classify the finite simple groups having such a triple.

There are a lot of examples and applications of both steps toward a new classification of the finite simple groups. Below we will mention a work which revised the earlier work of Janko, Smith, Gorenstein and Lyons classifying the finite simple groups all of whose 2 local subgroups are solvable.

1.4. **Revision started.** We will mention three works which should be regarded as a part of revision project for a new generation of classification.

The first is a work of Gomi, Hayashi, and Tanaka which gave a revised proof of the following theorem by Janko, Smith, Gorenstein and Lyons.

Theorem 6. *Let G be a finite simple group of characteristic 2 type all of whose 2 local subgroups are solvable. Then G is isomorphic to one of the following groups: $L_2(q)$, $Sz(q)$, $U_3(q)$ (where q is a power of 2), $L_2(p)$ (where p is a Fermat or Merssenne prime with $p \geq 5$), A_6, $L_3(3)$, M_{11}, $U_3(3)$, ${}^2F_4(2)'$.*

Using amalgam methods, Gomi, Hayashi, and Tanaka improved the earlier proof of Janko, Smith, Gorenstein and Lyons. In particular, with a few reasonable exceptions, they found the amalgams of the maximal 2 local subgroups, which made the proof conceptually easy to understand.

The second is a work of Bender and Glauberman which gave a revised proof of a part of the Odd Order Theorem by Feit and Thompson. Bender and Glauberman studied a part of local analysis, and gave new methods to investigate the structure of local subgroups of a simple group of odd order.

Now the third is a work of Suzuki who applied the new methods of Bender and Glauberman to investigate the structure of odd local subgroups of a simple group of even order. An odd local subgroup is, by definition, a normalizer of a (solvable) subgroup of odd order. Suzuki noticed that the methods of Bender and Glauberman to the imaginary groups (simple groups of odd order) were applicable to the real groups (simple groups of even order) as well. Inspired by the work of Suzuki, the author decided to study the structure of even local subgroups in the similar way. The next section is devoted to a kind of proposal toward the new approach to local analysis of finite simple groups.

2. NEW APPROACH

We begin with some definitions.

Definition 7. Let G be a finite group, and let π be a set of primes. A subgroup N of G is said to be a π *local subgroup* if it contains a nonidentity normal solvable π subgroup. A subgroup N of G is said to be π *separable* if every composition factor of N is either a π group or a π' group.

The above definition seems to be somewhat different from the one that we are used to, but we go along with our definition throughout the remainder of this note.

From now on, we will assume that G is a finite simple group. We need some more definition.

Definition 8. Let $\Gamma = \Gamma(G)$ be the *prime graph* of G, where the set of vertices $V(\Gamma)$ is the set of primes dividing the order of G, and two vertices p, q are *connected* if G has an element of order pq.

Let ϖ be a set of primes corresponding to a connected component of Γ. Depending on whether the prime 2 is contained in ϖ or not, we will call ϖ local subgroups either *even* local subgroups or *odd* local subgroups.

2.1. **Odd local subgroups.**

Let G be a finite simple group, and assume that $2 \notin \varpi$.

As stated in the previous section, Suzuki studied ϖ local subgroups, and obtained a new proof of the following theorem of Williams.

Theorem 9. *Let G be a finite simple group, and assume that $2 \notin \varpi$. Then G contains a nilpotent Hall ϖ subgroup which is isolated.*

We should note that the proof of Williams needed the classification of the finite simple groups, but that the proof of Suzuki did not. Looking closely at the work of Suzuki, the author thought that the similar method could be applied to the even local subgoups of finite simple groups.

2.2. **Even local subgroups.** Let G be a finite simple group, and let S be a Sylow 2 subgroup of G. Throughout the remainder of this note, assume that $2 \in \varpi$.

The first proposition gives information about the structure of a ϖ local subgroup.

Proposition 10. *Let N be a ϖ local subgroup of G, and let M be the largest normal ϖ separable subgroup of N. Suppose that $N \neq M$. Let E be a subgoup of G, where E/M is the product of minimal normal subgroups of N/M. Then E/M is a direct product of mutually* noniso-morphic nonabelian *simple groups.*

Definition 11. Let X be a subgroup of G containing S. The subgroup X is said to be S *irreducible* (or 2 *irreducible*) if S is contained in a unique maximal subgroup of X.

The next proposition gives information about the structure of a 2 irreducible subgroup.

Proposition 12. *Let X be an S irreducible subgroup of G. Then there are normal subgroups E and K of X such that $X = ES$, that $K \cap S = O_2(X)$, and that E/K is a direct product of mutually* isomorphic *simple groups.*

Combining Proposition 10 and Proposition 12, we obtained the following.

Theorem 13. *Let X be an S irreducible subgroup of G. Suppose that X is a ϖ local subgroup, and that X is not ϖ separable. Then there are normal subgroups E and K of X such that $X = ES$, that $K \cap S = O_2(X)$, and that E/K is a nonabelian simple group.*

We will apply Theorem 13 to a finite simple group of characteristic 2 type when ϖ consists of small number of primes.

In the rest of this note, let G be a finite simple group of characteristic 2 type, and let S be a Sylow 2 subgroup of G. Assume that $|\varpi| \leq 2$. Then we can classify the isomorphism classes of G as in the following forms.

Proposition 14. *One of the following holds:*

(1) *G is S irreducible, that is, S is contained in a unique maximal subgroup of G;*
(2) *Some (X, S, Y) has good properties (for example, both X and Y are S irreducible, $O_2(\langle X, Y\rangle) = 1$, and so on);*
(3) *Others.*

Proposition 15. *One of the following holds:*

(1′) *S is contained in a unique maximal 2 local subgroup of G;*
(2′) *Some (X, S, Y) has good properties (for example, both X and Y are S irreducible, $O_2(\langle X, Y\rangle) = 1$, and so on);*
(3′) *Others′.*

If one of the conditions (1), (2), (1′), (2′) holds, then the structure of G is determined by some existing general theory. To tell the truth, as of writing this manuscript, the author cannot fix what condition(s) should be placed in the third place in either form of the above propositions. It is quite subtle how the simple groups of our target should be scattered in the three categories. For example, the simple groups $L_3(3)$ and M_{11} with $\varpi = \{2, 3\}$ come into the part (3) of Proposition 14, while they come into the part (1′) of Proposition 15. Anyway, we will see almost the same simple groups coming out as the Theorem 6. It seems interesting whether the approach is applicable to simple groups with $|\varpi| \geq 3$.

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